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However, it would not seem wise that such a journal should be devoted entirely or chiefly to pedagogical discussions as such. It may, and often does, happen that the stimulus derived from reading an article not directly pedagogical in its nature may produce a reaction strongly beneficial on the pedagogical side. Such, for instance, may be the effect of articles like those of Professors Hedrick, Huntington and Coolidge in the present issue, and such without doubt has been the effect of Professor Cajori's "History of Logarithms" which has been running since January. Even those articles which appear too abstruse for a given reader may provide just the stimulus which he particularly needs, not only to insure his continued mental activity but to actually furnish a form of mental stimulus which may be essential to his successful work as a teacher. This is the reason for publishing in the MONTHLY such articles as those of Professor Lehmer in the May issue, of Dr. Miles in the April issue, and of Professor Dickson in the March issue. (See the reference to this question under Notes and News.)

Conclusion. While it is true that agitation does not necessarily mean progress, it is also true that there is seldom any progress without agitation. We confidently believe that the unprecedented activity among teachers of mathematics during the past decade has resulted, and will further result, in substantial progress. It should be a keen incentive to every teacher that he or she may have some active part in our nation-wide determination to reconsider the foundations of our teaching and to improve our methods wherever possible; and that in this effort we are allying ourselves with a world-wide movement toward the same end.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

BY FLORIAN CAJORI, Colorado College.

V. GENERALIZATIONS AND REFINEMENTS EFFECTED DURING THE NINETEENTH CENTURY.

THE GENERAL POWER AND LOGARITHM.

We have seen that the general theory of a^b, where both a and b are complex numbers, was outlined by L. Euler in his *Recherches sur les racines imaginaries des équations* of 1749, and that this paper failed to command the attention of mathematicians. We shall see now that three quarters of a century later the theory of the general power was elaborated by mathematicians of Germany, England, France and the Netherlands. At the opening of the nineteenth century this subject appeared difficult to many, as may be inferred from a paper of A. Q. Buée, in which the author contends that $\sqrt{-1}$ signifies perpendicularity,[1] and is finally led to the conclusion that $(\sqrt{-1})^n = \pm n(\sqrt{-1})$. The difficulty of the subject appears also in a paper of Argand who in 1813 ventured the statement that the expression $(\sqrt{-1})^{\sqrt{-1}}$ "would offer the simplest example of a

[1] *Philosoph. Trans.* for the year 1806, London, 1806, p. 67.

quantity irreducible to " the form $a + ib$.[1] Exception was taken to this statement by J. F. Français, professor in Metz, who pointed out that Euler had found $(\sqrt{-1})^{\sqrt{-1}} = e^{-1/2\pi}$, and showed that $(c\sqrt{-1})^{d\sqrt{-2}}$ is reducible[2] to $a + ib$. In the course of the next fifteen or twenty years sufficient familiarity was acquired with imaginaries to enable several mathematicians to grapple successfully with the theory of the general power. In the early history of the logarithms of positive numbers it was found surprising that logarithms were invented independently of exponents. Now another surprise is in store, namely that the theory of the general power is made to depend upon the theory of logarithms. All interpretations of a^b, where a and b are complex numbers, involve previously established results on logarithms. It thus appears that, historically, the logarithmic concept is the more primitive.

From the subtelty of general logarithmic theory arises the danger of its occasional recrudescence. We give an instance of this before we proceed to the general theory. Traces of confusion are found in an article by A. J. H. Vincent, professor at the Royal College at Reims—an article which possesses many points of merit. It touches upon the multiple nature of the logarithmic curve and its association with logarithmic theory. It is entitled, *Considérations nouvelles sur la nature des courbes logarithmiques et exponentielles.*[3] The author lets the variable x in $y = a^x$ take fractional values with even denominators, and obtains the usual two real branches. Vincent's article is discussed by George Salmon in his *Higher Plane Curves* (1879, page 286). Salmon accepts Vincent's idea that the curve has multiple branches. Vincent defines $1/n$ as the logarithm of all the numbers $a^{1/n}$, and concludes that, if a is positive, some $-$ numbers have real logarithms; if a is negative, some $+$ and some $-$ numbers have no logarithms, and so on. Vincent studies the discontinuities of his curves, but does not inquire into the consistency of his multiple-definition of logarithms. Peacock expresses himself as follows:

" The question of the identity of the logarithms of the same number, whether positive or negative, . . . has been frequently resumed in later times. The arguments in favour of the affirmative of this proposition, which were for the most part founded upon the analytical interpretation of the properties of the hyperbola and logarithmic curve, were not entitled to much consideration, in as much as they were not drawn from an analysis of the course followed in the derivation of the symbolical expressions themselves and from the principles of interpretation which those laws of derivation authorized."[4]

The Vincent article was criticized by J. P. W. Stein,[5] who took the view that every fractional index may be converted into one having an even denominator so that there would be a double number corresponding to *every* logarithm. Vincent was criticized also by D. F. Gregory who said:[6]

" It might, perhaps, have weakened his belief in the correctness of the results, if he had come to the conclusion, as he ought to have done, that the same logarithm corresponded to positive, negative, and impossible quantities. These last he seems quite to have overlooked."

[1] Gergonne, *Annales de Math.*, T. IV., 133–147.
[2] Same journal, Vol. IV, pp. 71–73.
[3] *Gergonne's Annales de math.*, Nismes, 1824 et 1825, T. XV., pp. 1–38.
[4] *Report of British Ass'n*, London, 1834, p. 266.
[5] Gergonne, *Annales de math.* T. XV, p. 231.
[6] *Cambridge Math. Jour.*, Vol. I, 1837, " On the Impossible Logarithms of Quantities."

A more serious recrudescence is seen in an article by L. C. Bouvier,[1] ex-officier de génie. In 1823 he derived the eulerian formulæ for $\log x$ and $\log(-x)$ from the relation $\log x = n(\sqrt[n]{x} - 1)$, $n = \infty$. Then, solving for x, he got $x = \left(1 + \frac{1}{n}\log x\right)^n$, *d'où, à cause de n infini, on peut conclure* $x = \left(1 + \frac{1}{n}\log x\right)^{n+1/k}$ $= \left(1 + \frac{1}{n}\log x\right)^n \cdot \left(1 + \frac{1}{n}\log x\right)^{1/k} = N\sqrt[k]{1}$, N being real and k any positive integer. Thus, not only is the logarithm of a number many-valued, but for any one logarithm the anti-logarithm is many-valued. For $k = 2$, $\log x = \log(-x)$. *Ces considérations nous semblent de nature à terminer, une fois pour toutes, le différend qui s'est élevé autrefois entre Euler et d'Alembert, sur la nature des logarithmes des quantités négatives.* Bouvier made no attempt to test the inherent consistency of his system. That there was inconsistency was claimed by J. P. W. Stein,[2] professor at the gymnasium at Trèves. Stein pointed out that Bouvier could not accept the eulerian formulæ, $\log x = r + 2m\pi\sqrt{-1}$, ($x$ positive) and $\log x = r + (2m+1)\pi\sqrt{-1}$ (x negative), and at the same time take $x = N\sqrt[k]{1}$. Starting into an untrodden path, Stein advances new arguments to show that Euler's formulæ are incorrect. From the relation $e^{r+z\sqrt{-1}} = y$, r being the real logarithm of y, z a number to be determined, Stein first deduces $e^{z\sqrt{-1}} = 1$ and from this obtains Euler's formulæ. *Mais il faut observer que, si e^r peut admettre plusieurs valeurs différentes,* say k values, then $e^r = y[\cos(2p\pi/k) + \sqrt{-1}\sin(2p\pi/k)]$. This result, combined with the eulerian values, yielded him $\log y = r + [2m - (2p/k)]\pi\sqrt{-1}$, $\log(-y) = r + [2m' + 1 - (2p'/k)]\pi\sqrt{-1}$. When k is even, he obtained $\log y = \log(-y)$. We see that, in Stein's system, the relation $x = \log y$ is no longer defined by the simple equation $e^x = y$, but by the equation $e^{x-(2p\pi/k)} = y$. The general expression for the logarithm of a number involves two independent arbitrary constants, m and p, instead of one only, as in the eulerian system. More perplexing is the circumstance that the number of values k which e^r takes, was seemingly not supposed to be the same in all cases or for all numbers. The author does not explain how, under these conditions, his system can yield a theory of logarithms that is general, consistent and useful. Certainly Stein did not move in the direction of Cauchy, who, as we shall see, by the introduction of principal values, aimed at greater simplicity and the inauguration of law and order.

Proceeding to the general theory of a^b, we observe that an author of very marked influence in the development of this theory was Martin Ohm, professor in Berlin. In 1811, when he was privat-docent at the university of Erlangen, he conceived the idea of writing the system of mathematics which appeared later under the title *Versuch eines vollkommen consequenten Systems der Mathematik*, Nürnberg, 1822–1852. The aim of the work is indicated by its title. It has been much criticized, but the part on the general power and logarithms is meri-

[1] Gergonne, *Annales de math. p. et appl.*, T. XIV, 1823–4, p. 275.
[2] Same journal, T. xv, p. 110.

torious. These topics are treated in the second volume, of which the second edition, Berlin, 1829, lies before us. The first edition appeared in 1823. The edition of 1829 is the one usually quoted by contemporary writers. Ohm gave an exposition of his theory of logarithms as early as 1821, in a Latin thesis which had probably a very limited circulation.[1] We have never seen it quoted anywhere. Ohm introduces the terms *complete equation* and *incomplete equation.* An equation is *complete* when both sides of it have the same number of values representing exactly the same expressions; it is *incomplete* when one or several (but not all) of the many-valued expressions on the right and left are the same.[2] We have pointed out that Euler had occasion to consider these two types of equations in his paper of 1749 on logarithms. After having developed the eulerian theory of logarithms Ohm takes up the general power,[3] "allgemeine Potenz," a^x, where both a and x are complex numbers, namely $a = p + qi$, $x = \alpha + \beta i$. Assuming e^z as always single-valued, and letting $r = \sqrt{p^2 + q^2}$, $\log a = Lr + (\pm 2m\pi + \varphi)i$, he takes $a^x = e^{x \log a} = e^{\alpha \cdot Lr - \beta(\pm 2m\pi + \phi)} \cdot \{\cos[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)] + i \sin[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)]\}$, where $m = 0, +1, +2, \cdots$ and L signifies the tabular logarithm. Thus the general power has an infinite number of values, but all are of the form $a + bi$. Ohm shows (1) that all of the infinite values are equal when x is an integer, (2) that there are n distinct values when x is a real, rational fraction s/n, (3) that some of the values are equal, though the number of distinct values is infinite, when x is real but irrational, (4) that the values are all distinct when x is imaginary. His idea of the irrational is partly explained by the statement that, if $x = s/n$ is irrational, then s and n are *unendlich grosse und nie angebbare Zahlen.* As illustrations of what his formula yields in special cases, Ohm shows that $i^i = (-i)^{-i} = e^{-(\pm 2d + \frac{1}{2})\pi}$, $d = 0, 1, \cdots$.

He inquires next, how the formulæ (A) $a^x \cdot a^y = a^{x+y}$, (B) $a^x \div a^y = a^{x-y}$ (C) $a^x \cdot b^x = (ab)^x$, (D) $a^x \div b^x = (a \div b)^x$, (E) $(a^x)^y = a^{xy}$ apply to the general exponent a^x, and finds that (A), (B) and (E) are incomplete equations, since the left members have " many, many more " values than the right members, although the right-hand values (infinite in number) are all found among the "infinite times infinite" values on the left; that (C) and (D) are complete equations for the general case.

Ohm next sets limitations upon the choice of values of *log a* and *log b*. When out of the infinite number of values of *log a*, some particular one, say α, is to be selected in the power $a^x = e^{x \log a}$, he indicates this by the notation $(a||\alpha)^x$. He writes similarly, $(b||\beta)^x$. With this understanding, we can write $x \log a + y \log a = (x + y) \log a$, and formula (A) becomes complete. The same is true of (B); while (C) and (D) remain complete equations under this restricted interpretation. It will be noticed that Ohm did not introduce the particular

[1] *De innumerosis novis logarithmorum generibus. Dissertatio qua ad examen solemne in gymnasio regio Thorunensi die XIV. Aprilis MDCCCXXI publice habendum omnes literarum cultores invitat Dr. Martinus Ohm, nonnullarum societatum literarum sodalis.* Berolini, Typis Haynianis.

[2] M. Ohm, *System d. Math.* 2, Theil, Berlin, 1829, p. 386.

[3] Ohm, *op. cit.*, 2. Theil, 1829, p. 412.

value of a^x, which is now called the "principal value." Aside from that his treatment of the general power is mainly that of the present time, except, of course, in the explanation of the irrational.

Ohm proceeds to the general logarithm, and he makes the novel and significant statement that "with the concept of the general power is given the concept of the general logarithm $b?a$, if by this is meant every expression x, such that one has $a^x = b$ or $e^{x \log a} = b$."[1] Notice his notation for the general logarithm, $b?a$. As x has an infinite number of values for each value of log a, we have here the appearance of two independent and arbitrary constants, as in the researches of Stein and in those of Graves, Hamilton, and others, to be discussed later. He says that since b and a are taken completely general, and a^x has an infinity of values, $b?a$ is wholly undetermined, unless we state for what value of *log* a the power a^x is to be taken. He adopts the special notation $b?(a \| \alpha)$, which means the logarithm of b to the base a, when $\log a = \alpha$. Ohm shows that $b?(a \| \alpha) = (\log b)/\alpha$ is a complete equation. If a is positive, say e, then Ohm's logarithmic system reduces to the familiar eulerian form. "And if a is not positive, then the general logarithm, as well as each of its special cases is different from every one of the logarithms hitherto defined and discussed."

It must be granted that Ohm has surpassed all his predecessors in the generality and fullness of discussion of the expression a^x, and that he is the first writer to successfully base the general theory of logarithms (having a complex number as a base) fully and unreservedly upon the theory of the general power a^x. But the theory of the general power, as we have seen, is not developed independently of logarithms; in fact it uses the theory of logarithms of complex numbers to the base e. Thus, the eulerian logarithms have answered as a step-ladder leading to the theory of the general power; the theory of the general power, in turn, has led up to a more general theory of logarithms having a complex base.

The first original research on the logarithms of complex numbers published in England appeared in 1829, in the *Philosophical Transactions*, from the pen of John Graves, then a young man of 23. Graves was a class-fellow of William Rowan Hamilton in Dublin. Hamilton repeatedly expresses his indebtedness to Graves and states that reflecting on Graves's ideas on imaginaries led, finally, to his invention of quaternions. Graves became a noted jurist. His paper of 1829 bears the title: "An attempt to rectify the inaccuracy of some logarithmic formulæ." The author modifies $l1$ by certain rather startling extensions, yielding $l1 = (2m'\pi\sqrt{-1})/(1 + 2m\pi\sqrt{-1})$. This is done by taking the base e not in its *arithmetical* form, but in its more general form $e^{1+2m\pi\sqrt{-1}}$, so that x is defined as the general logarithm of y, not by the equation $e^x = y$, but by the equation $e^{(1+2m\pi\sqrt{-1})x} = y = e^{\log y + 2m'\pi\sqrt{-1}}$, where log y means the real or tabular logarithm of y. Letting $y = 1$ gives $l1$ as written above. Thus, Graves (like Ohms) claimed that in the general expression for the logarithm there are two arbitrary and independent integers, m and m', instead of simply one, as given by Euler.

[1] Ohm, *op. cit.*, 2. Theil, Berlin, 1829, p. 438.

In a paper of subsequent date, published in the same volume of the *Philosophical Transactions* (1829), the Rev. John Warren of Cambridge contributed a paper which we noticed when we were considering the geometric representation of algebraic quantities. Warren arrived at some of the algebraic results of Graves. In June, 1832, Vincent published, at Lille, results identical in effect with the principal formulæ of Graves.[1]

For lack of explicitness in Graves's writings exception was taken to his views by Augustus De Morgan in his treatise on the Calculus of Functions, sections 158 and 245, published in the *Encyclopaedia Metropolitana*.

In 1833 George Peacock made a report on the recent progress of certain branches of analysis in which he touches upon eulerian logarithms,[2] and argues that even in that theory negative numbers may have real logarithms. For, consider $-a^m$ as originating from $(-1)(+a)^m$, then $l(-a)^m = (2r+2mr'+1)\pi\sqrt{-1} + m \log a$. If we suppose $m = \frac{1}{2}$, $r = 0$, $r' = -1$, we have $l(-\sqrt{a}) = \frac{1}{2}\log a = l\sqrt{a}$. The reader will see that the eulerian theory is here violated by the assumption that in $2mr'$ it is possible to take r' odd. As regards the work of Graves, Peacock thought that there was a fundamental error in his generalization, because he makes a *periodic* quantity the base of his system. Graves sent a defence to the British Association in 1834, also to the *Philosophical Magazine.*[3] The outcome of the discussion was that Graves withdrew the statement contained in the title of his first paper, to the effect that he was attempting to "rectify the inaccuracy" of the eulerian theory, while De Morgan admitted that if Graves desired to extend the idea of a logarithm so as to include not only the logarithms of the arithmetical form of the base, but also those of a more general form, there was no error involved in the process. In fact, De Morgan suggested a still further extension.[4] Logarithmic systems like that of Graves, in which there are two arbitrary and independent integers were also worked out by Sir William R. Hamilton.[5]

Without being aware, apparently, that De Morgan and Graves had come to an understanding, D. F. Gregory of Trinity College, Cambridge, published an able article "On the Impossible Logarithms of Quantities,"[6] in which he inquires "which is the correct result," that of Graves or of his opponents. Though admitting the possibility of both, Gregory considers the one with an arithmetical base as more expedient. Gregory also points out that Vincent, Peacock and Graves, each erroneously thought he had established that in certain cases there is a logarithm in common for positive and negative numbers. Gregory died at the premature age of 31, yet in his short scientific career he did much toward establishing the foundations of algebra.

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[1] *Report of the Fourth Meeting of the Brit. Assn.*, London, 1835, p. 524. We have not seen Vincent's publication of 1832.

[2] *Report of the Third Meeting of the British Ass'n*, London, 1834, p. 264.

[3] *Phil. Mag.*, Vol. VIII, 1836, p. 281.

[4] *Philos. Magazine*, Vol. IX, 1836, p. 252.

[5] "On Conjugate Functions or Algebraic Couples, etc.," *Transactions of the Royal Irish Acad.*, Vol. 17, Part II, 1835; *Lectures on Quaternions*, Dublin, 1853, preface, p. (12).

[6] *Cambridge Math'l Jour.*, Vol. I, 1837, p. 226.

Deep philosophic insight and logical power are evident in four papers *On the Foundation of Algebra,* published by Augustus De Morgan in 1842 and 1849. They were read before the Cambridge Philosophical Society in 1839, 1841, 1843 and 1844, respectively. From his first paper we quote the following:[1]

"If we define log x, or rather λx (reserving log x for the numerical logarithm of the length) to be any legitimate solution of $\epsilon^{\lambda x} = x$, it is plain that the logarithm of n inclined at an angle ν, (or of N) to the base b inclined at an angle β, (or B) is to be derived (avoiding ambiguity) from

$$(b\epsilon^{\beta\sqrt{-1}})^x = = n\epsilon^{\nu\sqrt{-1}},$$

or

$$\lambda_B N = = \frac{\log n + \nu\sqrt{-1}}{\log b + \beta\sqrt{-1}}.$$

This result is real when $\log n/\log b = \nu/\beta$; nor is it more surprising that an impossible quantity (hitherto so called) should have a possible logarithm, than that exponential operations not containing $\sqrt{-1}$, or not interchanging exponents of length and direction, should in certain cases enable us to pass from one line to another."[2]

It is worthy of note that in discussing the exponent $\sqrt{-1}$ on page 184, the author discloses, though somewhat vaguely, the notion of abstract groups and develops the cyclic group of order four. This paper was written by De Morgan in 1839, or some years before Cayley and William Rowan Hamilton came out with their researches on groups. It is one of the earliest, perhaps the earliest, of English references to group theory.

De Morgan shows that Graves's result is a special case of the above. In his second paper, De Morgan proceeds to the study of a^b. Reasoning geometrically as before and letting (r, ρ) be a line of r units inclined to the unit line at an angle ρ, he calls the *logometer* of (r, ρ) or $\lambda(r, \rho)$, the logarithm of that line, not only with respect to its length, but also the quantity of revolution by which it attained its present direction. From $e^{\pi\sqrt{-1}} = -1$ he gets $\pi = \lambda(-1)/(\sqrt{-1})$,

"a proposition which, not many years since, was one of the mysteries of analysis. It is now a very simple geometrical proposition: the first side means a line of π units laid down positively on the unit-line; the second side means the logometer of a negative unit turned back through a right angle. Now the logometer of a negative unit is a line of π units erected positively perpendicular to the unit-line: whence the identity of the two sides is manifest."[3]

These subjects are elaborated also in De Morgan's *Double Algebra,* London, 1849. He aimed to avoid the consideration of ambiguous values of symbols, "a thing for which there is no necessity."

"The more I think on this subject, the better satisfied do I feel, that the new algebra should have no symbols of double or multiple value whatsoever; that is, that the meaning of each elementary symbol should not be considered as complete, unless it expresses the amount of revolution from the unit line by which it is to be made to attain its direction, as well as that direction itself."[4]

He points out that the consequences of the full extension of a^b turn out to be capable of expression by the particular case in general use. "Whereas,[5] in the common system ϵ and $\epsilon^{\sqrt{-1}}$ are the logarithmic bases employed for ordinary and periodic magnitudes, we have, in the system above described, employed $e^{(m+n\sqrt{-1})^{-1}}$, and $e^{(\nu-\mu\sqrt{-1})^{-1}}$."

[1] *Trans. of the Cambridge Philos. Society,* Vol. 7, 1842, p. 186.

[2] The double sign of equality is used by De Morgan to indicate "that every symbol shall express not merely the length and direction of a line, but also, the quantity of revolution by which a line, setting out from the unit line, is supposed to attain that direction."

[3] *Trans. Cambr. Phil. Soc.,* Vol. 7, 1842, p. 294.

[4] De Morgan, *Trans. Cambr. Phil. Soc.,* Vol. 7, p. 296.

[5] *Trans. Cambr. Phil. Soc.,* Vol. VIII, Pt. II, p. 3.

An independent discussion of the generalized power $A^B = C$ is given by G. M. Pagani,[1] professor at the Catholic University of Löwen. He takes $A = \mu e^{\theta i}$, where the argument θ is called the *déterminatif* of A. The consideration of the *déterminatif* of a quantity is the only means, says the author, of avoiding paradoxes in algebraic analysis. He uses the notation $(a)^{m/n}$ to indicate all the n values of $a^{m/n}$. Assuming A and C as given, he calculates B. Letting $A = \mu e^{i\theta} = e^{l\mu + i\theta}$, $B = \alpha + i\beta$, he gets $C = e^{(l\mu + i\theta)(\alpha + i\beta)} = re^{i\phi}$ and finally $\alpha = (l\mu + \theta\phi) \div (l^2\mu + \theta^2)$, $\beta = (\varphi l\mu - \theta l\gamma) \div (l^2\mu + \theta^2)$. Here l means the tabular logarithm to the base e; θ and ϕ are each infinitely many-valued. Pagani points out that, if A is real and positive, so that $\theta = 2k\pi$, and if $\varphi = 2k\pi \cdot lr \div l\mu$, then the imaginary number $re^{i\phi}$ has a real logarithm. This conclusion is evidently erroneous. He says that his formulæ are *plus générales et par conséquent plus exactes* than those of Euler, since Euler's are obtained by letting $k = 0$. Pagani's paper was read before the Academy at Bruxelles on October 7, 1837. After the lecture Quetelet pointed out that J. S. Cerquero,[1] director of the observatory at San-Fernando, near Cadix, had reached similar results.[2] Later, Pagani prepared a note[3] on the limit of $(1+(x/n)^n$, x being complex, whereupon he was informed of the work of Graves in England on A^B. We shall see that the general power was treated also by Cauchy in 1847.[4]

Pagani's error, referred to above, arose from the tacit assumption that $(A^x)^y = A^{xy}$ is a complete equation. Ohm had shown that it is incomplete. A failure to pay attention to this matter leads to curious paradoxes. Th. Clausen of Altona gave such a paradox in 1827.[5] It was quoted by Peacock in his report.[6] When n is an integer, $e^{2n\pi i} = 1$, $e^{1+2n\pi i} = e$, hence also $e^{(1+2n\pi i)^2} = e = e^{1+4n\pi i - 4n^2\pi^2}$. Since $e^{1+4n\pi i} = e$, it would follow that $e^{-4n^2\pi^2} = 1$, which is absurd. In more condensed form the paradox is presented by E. Catalan[7] thus: $e^{2m\pi i} = e^{2n\pi i}$, where m and n are any two distinct integers. Raising both sides to the power $i/2$, we have the absurdity, $e^{-m\pi} = e^{-n\pi}$.

The reader will notice that $(e^{2m\pi i})^{i/2} = e^{-m\pi}$ is an incomplete equation; all values of the right member are values of the left member, but not *vice versa*. Let $((a))^x$ signify all the values of the general power. Since $((a))^x = e^{x \log a + 2p\pi i}$ is a *complete* equation, where $\log a = la + 2q\pi i$, $la =$ tabular logarithm, $\pm p = 0, 1, 2, \cdots$ and $\pm q = 0, 1, 2, \cdots$, we obtain the following complete equations $((\,((e))^{2m\pi i}))^{i/2} = ((e^{2m\pi i \log e + 2p\pi i}))^{i/2} = e^{-m\pi - p\pi - 2mq\pi^2 i}$. When $p = q = 0$, we have the particular value e^{-mx}. Similar results are obtained for $(e^{2n\pi i})^{i/2}$. It follows that $(e^{2m\pi i})^{i/2} = (e^{2n\pi i})^{i/2}$ is an incomplete equation, in which the

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[1] *N. mémoires de l'academie royale des sciences et belles-lettres de Bruxelles*, T. XI, Bruxelles, 1838, pp. 1–11.

[2] We have been unable to find any other reference to Cerquero's research.

[3] *Bulletins de l'académie r. d. scien. et b. l. de Bruxelles*, T. 6 (*1re partie*), *Bruxelles*, 1839, p. 256.

[4] Cauchy, *Exercices d'analyse et de phys. math.*, T. IV, 1847, p. 255.

[5] *Crelle's Journal*, Vol. 2, Berlin, 1827, pp. 286–287.

[6] *British Ass'n Report*, London, 1834, p. 347.

[7] *Nouv. Ann. de Math.*, 2e S., Vol. 8, 1869, p. 456. See also Vallès in the same journal, 2e S., Vol. 9, 1870, p. 20.

particular value e^{-mx} on the left is not equal to the particular value e^{-nx} on the right. This last equation becomes complete under the restrictions $m + p = n+p'$ and $mq = nq'$, where the integers p' and q' are used in the complete value of the right member in the same way as p and q were used on the left. De Morgan draws upon the notion of direction of lines to explain paradoxes of this character.[1] He reminds us that such fallacies have been seriously proposed as arguments against the use of imaginary quantities.

The generalizations of logarithms leading to systems having periodic or complex bases gave rise to formulæ of such complexity that the generalized system failed of adoption. The machinery was too intricate for general use. The theory of the general power a^b, a and b being complex numbers, was established satisfactorily with the aid of the eulerian logarithms to the base $e = 2.718$. No real advantage grew out of the new logarithmic systems. If in $a^x = c$, the complex numbers a and c are given and x is to be found, the eulerian logarithmic formulæ are sufficient for the solution. Thus it happened that the general logarithmic systems of Ohm, Graves, Vincent, Warren, De Morgan, Gregory, Hamilton and Pagani failed of recognition as useful mathematical inventions.

However, one result of the study of the general power and of logarithms having a number $a + ib$ as a base was of importance; this study led to the conclusion that no possible combination of real and ordinary complex numbers gives rise to new forms of numbers. De Morgan frankly admits that he had expected some "new *imaginary* or *impossible* quantities."[2] But the processes of addition and subtraction, multiplication and division, involution and its two inverses, evolution and logarithmation, constitute a closed group of operations; they complete the cycle of operations in the algebra of ordinary complex numbers. In the seventeenth century the logarithmation of negative and complex numbers was not recognized; hence the cycle of operations was not a closed one and the algebra of that time was not complete.

Similar researches were carried on for quaternions. We have seen that William Rowan Hamilton was in close touch with the work of J. T. Graves and De Morgan. He was familiar with the investigations of Ohm. Like Cauchy and others, Hamilton recognized the great inconvenience arising from the multiplicity of logarithmic values. ". . ., but it has been my object, in the present theory, to preclude, so far as I could, that indeterminateness by *definition*."[3]

A difficulty in Hamilton's development of general quaternion theory—his algebra of space—is described by him as follows:

> In the present theory of diplanar quaternions, we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and easier case of complanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value.[4]

More recently this difficulty has received the attention of Alexander Mac-

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[1] De Morgan, *Double Algebra*, p. 136.
[2] *Trans. Cambr. Phil. Soc.*, Vol. VIII, Pt. II, 1844, pp. 3, 4.
[3] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, pp. (15), 556.
[4] W. R. Hamilton, *Elements of Quaternions*, London, 1866, Section 11, p. 386.

farlane. He points out that in the algebra of the plane there is no need of specifying the axis of rotation for a circular angle, such as e^{ib}. It is different in space. When angles on a sphere are considered, the axis must be inserted, and also $\pi/2$ for the indefinite i.[1] If by α^a we mean a radians round the axis α, α^1 means 1 radian round α. Here $\log \alpha^1 = \alpha^{\pi/2}$ and $\log \alpha^a = a \log \alpha^1 = a\alpha^{\pi/2}$. Whence $\alpha^a = e^{a \log_e \alpha^1} = e^{a\alpha^{\pi/2}}$. Now Hamilton adopted the principle which, in Macfarlane's notation is $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta + \sin\alpha\beta[\alpha\beta]^{\pi/2}$, where $[\alpha\beta]$ signifies the unit which is normal to α and β. Macfarlane, on the other hand, adopts the principle $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta - \sin\alpha\beta\,[\alpha\beta]^{\pi/2}$. By this modification Macfarlane is able to establish for his space-algebra the simple theorem which Hamilton could not derive from his assumption, $\log(e^{a\alpha^{\pi/2}} \cdot e^{b\beta^{\pi/2}}) = a\alpha^{\pi/2} + b\beta^{\pi/2}$.

A SIMPLE FORMULA FOR THE ANGLE BETWEEN TWO PLANES.

By EDWARD V. HUNTINGTON, Harvard University.

Let us consider the familiar problem of finding the area of a surface by the methods of the integral calculus.

Let S be any closed portion of a surface whose equation is $z = f(x, y)$, and let ΔS be an "element" of the surface at any point P. Then, as is well known,

$$\Delta S = \frac{\Delta S'}{\cos\gamma},$$

where $\Delta S'$ is the projection of ΔS on the xy-plane, and γ is the angle between the xy-plane and the tangent plane to the surface at P. The area of S is therefore given by

$$S = \int\int \frac{dS'}{\cos\gamma},$$

where the integration is to be extended over the whole of the plane area S' which is the projection of S on the xy-plane.

The problem of finding the area S thus reduces to a simple problem in double integration, *provided the value of γ at every point of the surface is known.*

In the ordinary treatment of the subject in the text-books, the derivation of the formula for γ requires a considerable knowledge of analytical geometry of three dimensions, including the equation of a plane, the equations of a straight line normal to a plane, the method of finding the direction cosines of a line when its equations are given, the equation of a tangent plane, etc.

The object of this note is *to show how the angle γ may be found directly,* by very elementary methods. The formula obtained, in the easily-remembered form here given, and the methods of derivation are so simple that it is hardly possible that they are new; and yet I do not find them in any of the current text-books.

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[1] A. Macfarlane, "A vector-analysis as generalized algebra," *International Congress of Mathematicians*, Cambridge, August, 1912; "Account of researches in the algebra of physics," *Journal of the Washington Acad. of Sci.*, Vol. II, Nos. 14, 15, 16, 1912.